On the accuracy of a calibrated DistoX

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The a formal study of the properties of a DistoX-like device have not been much investigated. Heeb discusses of the accuracy of the calibration algorithm from a practical point of view. In this short note I want to address the issue of the accuracy of a ideal well calibrated DistoX. The idea is to demonstrate that an "ideal" DistoX that is "well calibrated" is accurate, ie, it measures the true directions. A line of thought to attack the problem on theoretical ground is here suggested, but the details are not worked out, and there might be mistakes.

Definition a <u>DistoX</u> is a device that measures the 3D vector between two 3D points as triplets of distances, azimuth and inclinations.. The distance is measured with no error (total precision).

The measure axis is called **X** axis.
The values measured between the two points A and B are

- d(AB) distance between A and B
- a(AB) azimuth of the point B seen from A
- c(AB) inclination of the point B seen from A

AB denotes the vector from A to B.

Definition a DistoX is <u>calibrated</u> if the following two conditions are valid:
(1) for any measurement between two points A, and B, azimuth and inclination are independent of the roll angle, ie of the angle of rotation of the DistoX around the measure axis.
(2) for any pair of 3D points, A and B, the measurements AB and BA satisfy:

- (2a) d(AB)=d(BA),
- (2b) a(AB)=(a(BA)+180) mod(360),
- (2c) c(AB)=-c(BA)

(3) for any triplet of 3D pointsA, B, and C, the triangle ABC closes with zero error:

- (3a) d(AB) sin(c(AB)) + d(BC) sin(c(BC)) + d(CA) sin(c)CA)) = 0
- (3b) d(AB) sin(c(AB)) cos(a(AB)) + d(BC) sin(c(BC))cos(a(BC)) + d(CA) sin(c(CA)) cos(a(CA)) = 0
- (3c) d(AB) sin(c(AB)) sin(a(AB)) + d(BC) sin(c(BC)) sin(a(BC)) + d(CA) sin(c(CA)) sin(a(CA)) = 0

With vector notation thesse two conditions are

- (1) **AB** is independent of the roll
- (2) **AB** + **BA** = **0** for any A, B
- **(3) AB + BC + CA = 0** for any A, B, C

The *accuracy result would be* that a calibrated DistoX measures the true azimuth and inclination. Here i try to prove it.

The space of DistoX orientation in 3D space is a 2-sphere (azimuth and inclination) times the roll 1-sphere: a spherical shell with identified outer and inner points. The triplets (a,c,r), azimuth, clino and roll, describe the points in this space. The orientation (a,c,r) is mapped to (a',c',r') by the measurment process. The projection on the 2-sphere gives (a'.c') = **F**(a,c,r).

[1] The first step to obtain the accuracy result is to recognize **F** as a function from the 2-sphere. This is a consequence of the calibration condition (1), the independent of the roll. Therefore **F** is a function from the 2-sphere into itself.

The DistoX would not be accurate if **F**(a,c) is different from (a,c). The inaccuracy is

$$\mathbf{V}(a,c) = \log_{(a,c)} \mathbf{F}(a,c)$$

V(a,c) lies in the tangent plane to the 2-sphere at **P**=(a,c) directed along the great circle joining **P**=(a,c) to **F**(a,c), and with length the arc-length between **P** and **F**(a,c). Approximately **V**(a,c) = **F**(a,c) - **P**.
V is a vector field on the 2-sphere. If the DistoX is accurate **V**(a,c) = 0 for any (a,c).
Each shot measurement is the sum of the true vector and the inaccuracy vector,

$$\mathbf{s'} = \mathbf{s} + \mathbf{v}$$

[2] The second step is to show that the vector field V is <u>continuous</u>. Intuitively this follows from the way the device orientation is measured by the DistoX, namely from the measurements of the real vectors **G** and **M**. The three components of these vectors in the ideal frame of reference of the DistoX changes continuously with the device orientation, and the calibration is a linear map, thus continuous.

[3] The third step is to use Brouwer's theorem which states that a vector field over the 2-sphere must vanish at one point $\mathbf{P_0}$ at least: $\mathbf{V}(\mathbf{P_0}) = 0$. The point $\mathbf{P_0}$ is called the *pole.*

[4] The calibration conditions (2) and (3) entails some constraints on the vector field **V.**
The fourth step is an immediate consequence of (2) is that for any point **P**, **V**(-**P**) = - **V**(**P**) where the opposite of **P**=(a,c) is **-P**=(a+180 mod 360, -c).

Therefore V vanishes also at the opposite point of $\mathbf{P_0}$ (called the *antipole*): $\mathbf{V}(-\mathbf{P_0}) = 0$. The *polar line* is the diameter joining the pole and the antipole. Maximum semicircles through the pole and the antipole are called *meridians*. The maximum circle orthogonal to the polar line is the *equator*. In the following we use "polar coordinates" (a,c) with c=+90 for the pole and -90 for the antipole. The a=0 meridian is arbitrary.

[5] The fifth step is to consider diametrically opposite points **R** and **S** on a circle in a plane perpendicular to the polar line, and show that **V**(**R**) = - **V**(**S**) and both vectors lie in the plane of the circle. If c is the inclination of the points (on the 2-sphere), the result follows considering the triangle $[\mathbf{R}, \mathbf{S}, -2\sin(c)\mathbf{P_0}]$.

[6] The sixth step is to relate **V**(**R**) to **V**(**Q**) where **Q** is the equatorial point on the meridian passing through **R**: **V**(**R**) = cos(c) **V**(**Q**). To get this we use the triangle [**S**, -**R**, 2 cos(c) **Q**].

[7] Next (step seven) we should show that the vector field **V**(**Q**) on the equator, lies in the equatorial plane, and has constant length, ie, $|\mathbf{V}(\mathbf{Q_1})| = |\mathbf{V}(\mathbf{Q_2})|$ for any $\mathbf{Q_1}$, $\mathbf{Q_2}$ on the equator. After we have proved this we can write

$$\mathbf{V}(a,c) = w_0 \cos(c) [-\sin(a)\, \mathbf{u_x} + \cos(a)\, \mathbf{u_y}] = w_0 \cos(c)\, \mathbf{u_a}$$

where $\mathbf{u_x}$ and $\mathbf{u_y}$ are the unit vectors in the equatorial plane.

[8] If we can show that $w_0 = 0$, we have completed the proof of the accuracy result. We cannot find a 3D loop around which the integral of **V** is non-zero unless $w_0 = 0$: for any closed 3D loop the integral of **V** vanishes. Therefore to prove the final step we must resort to roll invariance.

Consider the direction in the 3D horizontal plane perpendicular to the magnetic North. The true G and M lie in a plane orthogonal to this direction. If the DistoX does not measure correctly East, but **E**+w**u**, then rotation the DistoX by 180° around the laser axis it measures **E**-w**u**. Roll invariance is possible only if w=0. If East is the direction of the pole of the 2-sphere, we consider the measurements in the direction of North.