Survey accuracy and DistoX

marco corvi - marco.corvi@gmail.com
Apr. 22, 2018

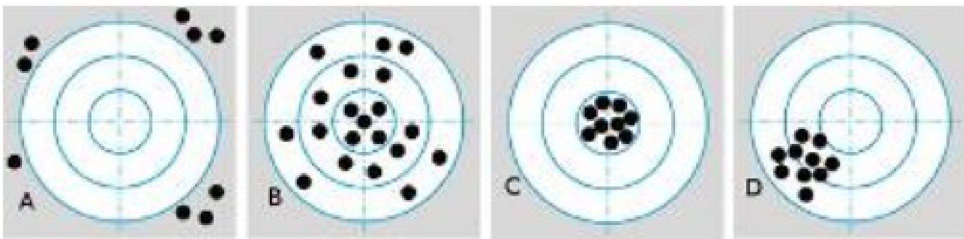
Like in any other measurement work, we want to come back with an accurate set of data from a cave surveying trip. Therefore it is important to know if the instrument we are using can provide accurate data and in what range we can trust the data.

DistoX is a cave measurement instrument widely used currently [1,3]. Since Leica guarantees the distance measurement accuracy of DistoX, in this note we address the accuracy in measuring directions of a calibrated DistoX. In this note it is assumed that the distance measurements are very precise and accurate.

The main result of this work is that a DistoX that satisfies the calibration tests (described below) measures accurate values.

Accuracy and precision

When we measure a quantity, an *observation* refers to a reading of the value of the quantity. Themeasuring process usually involves several observations, and the result is the set of *measured values*. *Accuracy* is the degree by which the measured values agree to the real value [5, 6]. *Precision* is the uncertainty in the measured values. Accuracy relates the measured values and the real value. When we state "$x = 10.2 \pm 0.5$" we mean that the real value lies within the interval [9.7, 10.7] (with some high probability, usually 97%). The precision is the uncertainty with which we can take an observation. For example, if the instrument has the smallest reading unit of 0.1, we might observe "$x_1 = 10.4 \pm 0.05$", "$x_2 = 10.1 \pm 0.0.5$", etc.



In the above figure, A has a low accuracy and a low precision, B is accurate but not precise, C is both accurate and precise, D is precise but inaccurate.

In practice, the precision is the uncertainty that we get when we make a reading. It is not only due to the instrument, but also due to the measuring process, ie, how the operator takes the shot, including positioning the instrument on the station and aiming at the target.

DistoX calibration checks

The calibration of a DistoX consists of finding the (approximate) transformation that maps the values measured by the on-board sensors to the values of the gravity and magnetic field in the frame of reference of the device (with X-axis aligned to the laser beam, see figure). The calibration procedure requires taking shots at various directions covering all orientations. For each direction a group of four shots at different angles of roll are taken [2].

The quality of a calibration is usually checked by examining the consistency of shots from A to B taken at different angles of roll, and the closure error of sequences of shots that form a closed cycle (loop).

Thus an ideally-calibrated DistoX will satisfy

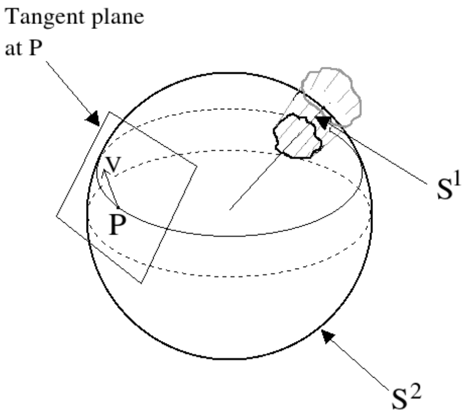
- shots A-B at different roll have the same azimuth and inclination values,
- there is no closure error for any loop

A DistoX is ideally-calibrated if it satisfies the following calibration checks:

1) for any shot A-B, azimuth and inclination are independent of the roll angle,
2) for any pair (A,B), the azimuth and inclination value of A-B are opposite to those of B-A,
3) for any triple (A,B,C) the loop A-B, B-C, C-A closes with zero error.

The main result of this note is that a well-calibrated DistoX is accurate in measuring directions, ie, the measured values of azimuth and inclination agree (within the measurement uncertainties) with the real azimuth and inclination.

This result has an important practical implication. As the accuracy of distance measurements is guaranteed by Leica to be of the order of a few millimeters [4], using the "good-calibration" checks one can verify if a DistoX is properly calibrated, and can estimate the uncertainty on the data due to the instrument.

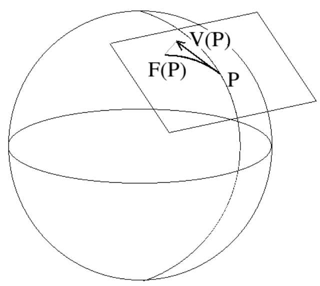


The ideal DistoX

The ideal DistoX has infinite precision and perfect calibration. Infine precision means that the reading uncertainty is always zero. Perfect calibration means that the data within each calibration data group differ only by the value of roll, and the calibration algorithm ends in one iteration with zero residual error. The ideal DistoX satisfies the calibration checks perfectly.

The space of DistoX orientation is a 2-sphere times the roll 1-sphere (locally $S^2 x S^1$), a shell around the 2-sphere with identified outer and internal points along each radius.

We use coordinates (a,c,r), azimuth, clino and roll, to describe points in this space. The orientation (a,c,r) (real value) is mapped to a point (a',c',r') by the measuring process, and the projection on S^2: (a',c') = $\mathbf{F}(a,c)$, is the



measured value. Due to calibration check (1) **F** does not depend on *r*. The inaccuracy field **V**(*a*,*c*) is the difference between the measured value and the real value,

$$\mathbf{V}(a,c) = \log_{(a,c)} \mathbf{F}(a,c)$$

V(*a*,*c*) lies in the tangent plane at P=(*a*,*c*): its direction is along the great circle joining P to **F**(*a*,*c*) and its length is the arc-length between P and **F**(*a*,*c*). Approximately **V**(*a*,*c*) = **F**(*a*,*c*) - P .

The inaccuracy field

In the following we use the notation of an uppercase letter, eg. P, for the coordinate pair (*a*,*c*). **V**(P) is the inaccuracy field: a vector field in the tangent bundle of the DistoX space. The tangent planes are embedded in R^3, and the vectors in the tangent plane are considered as vectors in R^3. The opposite direction of P is -P.

We denote a sequence of shots by a series of vectors in square brackets, each vector is composed by its length and its direction, mapped to a point on S^2. The length is omitted if it is 1. For example [A+bB+cC] denotes the sum of vectors $\mathbf{s'} = \mathbf{u'}_A + b\,\mathbf{u'}_B + c\,\mathbf{u'}_C$. Each shot can be splitted into the real value and the inaccuracy value, therefore the sum of shots can be splitted into the sum of the real values and the sum of the inaccuracy values, **s'** = **s** + **v**, where $\mathbf{s} = \mathbf{u}_A + b\,\mathbf{u}_B + c\,\mathbf{u}_C$, and **v** = **V**(A) +b **V**(B) + c **V**(C). If the sum of the real values is zero, **s**=0, the sequence of shots is a loop.

Claim 1. **V**(-Q) = -**V**(Q)

Proof. Consider the loop [Q + (-Q)], consisting of the shot A-B, the point Q on S^2, and B-A, corresponding to the point -Q. The claim follows from **V**(Q)+**V**(-Q) = 0.

Claim 2. There is at least one point, P_0, on S^2 with $\mathbf{V}(P_0) = 0$

Proof. This result follows from Brouwer's therorem [6].

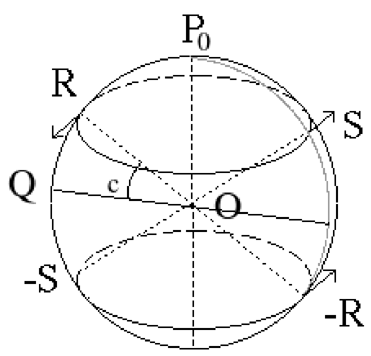
P_0 is called *pole*. By claim (1) **V** vanishes also at the opposite point, the *antipole*, $\mathbf{V}(-P_0) = 0$; The line beween pole and antipole is the *polar* line, and the maximum circle orthogonal to the polar line is the *equator*. Maximum circles through the pole and the antipole are called *meridians*. In the following we use polar angles, azimuth and clino, in this "reference system"; they are not related to the azimuth and clino in 3D space.

Claim 3. Let's consider diametrically opposite points, R and S, on a circle in a plane perpendicular to the polar line, then

V(R) = - **V**(S),

and both lie in the plane of the circle.

Proof. Take the loop [R + S - 2 sin(*c*) P_0], where *c* is the inclination of R (angle between RO and the plane of the equator).



Since **V**(R) + **V**(S) - 2 sin(c) **V**(P_0) = 0, we obtain **V**(R) = - **V**(S).
Now **V**(R) lies in the tangent plane at R and **V**(S) in the tangent plane at S. There is only one direction common to the two planes: the normals to the planes are OR and OS, and the common direction is their cross product OSxOR, which lies in the plane of the circle.
Therefore **V**(R) lies in the plane of the circle.

Claim 4. **V** at a point R on the circle at inclination *c* is cos(*c*) times **V** at the equatorial point Q on the meridian that passes through R.

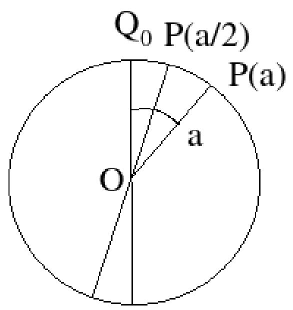
Proof. Consider a point Q on the equator and the two points S and -R on two circles at inclination *c* and -*c*, respectively, and in the meridial plane that contains the polar line and Q. From the loop [S + (-R) + 2 cos(*c*) Q] we have

$$\mathbf{V}(S) + \mathbf{V}(-R) + 2\cos(c)\,\mathbf{V}(Q) = 0$$

therefore **V**(R) = cos(*c*) **V**(Q).

Claim 5. The vector field **V** on the equator lies in the equatorial plane and has constant length.

Proof. Let's restrict to the equatorial plane and consider the vertical (out of plane) component of **V**, denoted v, and the component in the equatorial plane, **w**.
On the equator there must exist a point Q_0 with zero vertical component, v(Q_0)=0, because of claim (1) and continuity.
Consider a point P(*a*) on the equator, at an angle *a* with respect to Q_0 and the loop [Q_0 + P(*a*) + (-2cos(*a*/2)P(*a*/2))]. We write the shorthand v(*a*) and **w**(*a*) for v(P(*a*)) and **w**(P(*a*)), respectively, and w(*a*) for the length of **w**(*a*). For the vertical component we have v(*a*) - 2 cos(*a*/2) v(*a*/2) = 0 which has solution (v_{90} is a constant)



$$v(a) = v_{90}\sin(a)$$

As for the components in the equatorial plane we have along the axis OQ_0

$$\sin(a)\,w(a) - 2\cos(a/2)\sin(a/2)\,w(a/2) = 0$$

Therefore

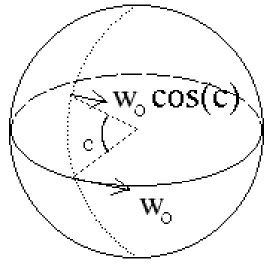
$$w(a) = w_0 = w(Q_0)$$

The equation for the orthogonal axis is then identically satisfied,

$$w_0 + \cos(a)\,w(a) - 2\cos^2(a/2)\,w(a/2) = w_0\,[\,1 + \cos(a) - 2\cos^2(a/2)\,] = 0$$

Taking into account claim (3) and continuity v_{90} = 0, thus v(*a*) = 0. Therefore **V** lies in the eqautorial plane and is of constant length, w_0.

Interlude 6.

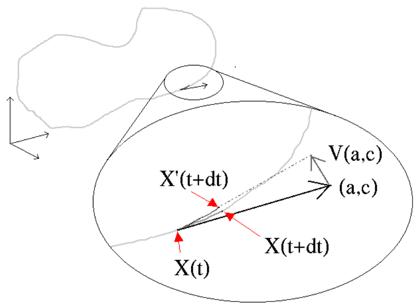


So far we have identified the inaccuracy field

$$\mathbf{V}(a,c) = w_0\cos(c)\,[\,-\sin(a)\,\mathbf{u}_x + \cos(a)\,\mathbf{u}_y\,] = w_0\cos(c)\,\mathbf{u}_a$$

where $\mathbf{u}_y$ is the unit vector along OQ_0 and $\mathbf{u}_x$ the orthogonal one. Consider a survey loop, a curve in 3D space. Parametrized by the arc-length *t*, the loop consists of infinitesimal segments of length dt and direction with azimuth *a*(*t*) and inclination *c*(*t*). The closure of the loop implies that the integrals over the closed loop of the infinitesimal displacements in the X, Y, and Z directions are zero,

$$\int \cos(c(t)) \cos(a(t))\, dt = 0$$
$$\int \cos(c(t)) \sin(a(t))\, dt = 0$$
$$\int \sin(c(t))\, dt = 0$$



Therefore the integral of the inaccuracy field along the loop

$$\int [\, - w_0 \cos(c) \sin(a)\, \mathbf{u}_x + w_0 \cos(c) \cos(a)\, \mathbf{u}_y \,]\, dt$$

vanishes due to the first two loop-closure integrals.
Therefore the loop closure alone is not sufficent to establish that the inaccuracy field vanishes. Note that the above inaccuracy field is C^{∞}.
In particular around the pole the vector field is $\mathbf{V}(x,y) = - y\, \mathbf{u}_x + x\, \mathbf{u}_y$

To show that $w_0 = 0$ we resort to use roll invariance.

Claim 7. $w_0 = 0$

Proof. Consider the point $\mathbf{n} = (1,0,0)$, corresponding to a=0, c=0. Suppose that $\mathbf{V}(0,0) = w_0\, \mathbf{u}_y$ is not null. A rotation about the axis of the DistoX amounts to the multiplication by the rotation matrix $R\mathbf{n}(t)$ with axis $\mathbf{n}$ and angle *t*. Thus $R\mathbf{n}(t)\, \mathbf{V}(0,0) = w_0\, (\cos(t)\, \mathbf{u}_y + \sin(t)\, \mathbf{u}_z)$. This has non-null z component unless $w_0 = 0$.

Theorem

A well-calibrated DistoX is accurate in measuring the directions, ie, $\mathbf{V}(R) = 0$

Conclusions

Although described using DistoX, this result is quite general and it applies to any measuring device that satisfies the calibration checks and is accurate in measuring the distances.

As a counterexample we take the "set of tape and Suunto compass and clino" [8]. In this case the device's intrinsic frame of reference has the X axis along the line of sight, the Y axis orthogonal to it in the horizontal plane, and the Z axis to form a right-handed triplet. The Y axis is chosen so that the Z axis is downwards. The compass-clino device is not "roll invariant". Therefore the compass-clino pair can be affected by the inaccuracy field described in [8]. Infact, a clino miscalibration is detected (and corrected) by comparing the inclinations of A-B and B-A, but the accuracy tests cannot detect a miscalibration of the compass.

The analysis of the extent to which uncertainties in the calibration checks affect the accuracy of the measurents remains open for future work.

References

1. B. Heeb, *Paperless caving - An electronic cave surveying system*, Spelunca Memoires 33 2008 130-133
2. B. Heeb, *An all-in-one electronic cave surveying device*, CREG Journal 72 2009 8-10
3. B. Heeb, *The next generation of DistoX cave surveying instruments*, CREG Journal 88 2014 ...
4. Leica Geosystems http://lasers.leica-geosystems.com/x310
5. Flatiron Surveying Inc., *Accuracy and precision*, www.flatsurv.com/accuprec.htm
6. K.E. Foote, D.J. Huebner, *Errors, accuracy, and precision*, www.colorado.edu/geography/gcraft/notes/error/error.html
7. Eisenberg, Murray, Guy, Robert, *A proof of the hairy ball theorem*, The American Mathematical Monthly, 86 1979 571-574
8. G. Veni, *How to use the Suunto*, Compass and Tape 1 1983 4-5